Maths for Computing Tutorial 3

1. Prove that if a is rational and ab is irrational, then b is irrational.

Solution: We will prove it by contradiction. Assume that a is rational and ab is irrational and b is rational. But if a and b are rational, then ab is also rational, which is a contradiction as we have assumed that ab is irrational. Hence, if a is rational and ab is irrational, then b is irrational.

2. Prove that $\sqrt{3}$ is irrational.

Solution: Assume $\sqrt{3}$ is rational. Then,

$$\sqrt{3} = a/b, \tag{1}$$

where $a, b \in \mathbb{Z}^+$ such that a and b have no common factors.

Squaring (1) on both sides and rearranging gives us $3b^2 = a^2$. This implies that 3 is a factor of a^2 . Since 3 is a prime, 3 should also a be a factor of a, i.e., a = 3k. Replace a with 3k in $3b^2 = a^2$. We will get $b^2 = 3k^2$, which implies that b has 3 as a factor using similar reasoning as we did for a. But this is a contradiction. In (1) we assumed that a and b have no common factors, but now we have shown that they have a common factor 3. Hence, $\sqrt{3}$ is irrational.

3. Prove that if *n* is a perfect square, then n + 2 is not a perfect square.

Solution: We will do a proof by contradiction. Assume that *n* and n + 2 are perfect squares.

Then,

$$n = k^2$$
 (1)
 $n + 2 = l^2$ (2)

where k and l and non-negative integers.

Subtracting (2) from (1) gives us:

$$2 = l2 - k2$$
$$2 = (l + k)(l - k)$$

This implies that (l + k) is a divisor of 2. But among all possible values of l and k where l + k is a divisor of 2, none gives $l^2 - k^2 = 2$. Hence, we have found a contradiction. Therefore, if n is a perfect square, then n + 2 is not a perfect square.

4. Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a^2 + b^2 = c^2$, then at least one of a and b must be even. **Solution:** We will do a proof by contradiction. Assume that $a^2 + b^2 = c^2$ and both a and b are odd. Let a = 2k + 1 and b = 2k' + 1 and replace a and b in $a^2 + b^2 = c^2$.

$$a^{2} + b^{2} = c^{2}$$

$$(2k + 1)^{2} + (2k' + 1)^{2} = c^{2}$$

$$4k^{2} + 4k + 1 + 4k'^{2} + 4k' + 1 = c^{2}$$

$$4(k^{2} + k + k'^{2} + k') + 2 = c^{2}$$

$$2.(2l + 1) = c^{2}$$

This implies that c^2 is even and hence c is also even. Let c = 2j for some $j \in \mathbb{Z}$ and replace c with 2j in $2 \cdot (2l + 1) = c^2$.

$$2.(2l + 1) = (2j)^{2}$$
$$(2l + 1) = 2j^{2}$$

The above equation is a contradiction as LHS is odd and RHS is even. Hence, if $a^2 + b^2 = c^2$, then at least one of *a* and *b* must be even.

5. Prove that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

Solution: (The flow of the proof is different from the one we discussed. The one given below was shown to me by one student in the second session.)

A long exhaustive proof is possible but there is a better proof by contradiction. Suppose we can arrange first 10 positive integers around a circle such that no three consecutive integers' sum \geq 17. Suppose the 10 integers are named as $a_1, a_2, a_3, \ldots, a_{10}$ such that a_{i+1} has a_i before it and a_{i+2} after it (a_1 is after a_{10}). Since no three consecutive integers' sum is more than 17, we can say that $a_i + a_{i+1} + a_{i+2} \leq 16$, for $i \in [1,8]$, $a_9 + a_{10} + a_1 \leq 16$ and $a_{10} + a_1 + a_2 \leq 16$. If we sum these eight inequalities we will get $3(a_1 + a_2 + \ldots + a_{10}) \leq 160$ as every a_i appears in three inequalities, which is a contradiction as $a_1 + a_2 + \ldots + a_{10} = 55$.

6. Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros.

Solution: It's a proof by contradiction. Suppose we reach nine zeros for the first time by repeatedly applying the given operation. Now, we can get nine zeros only if in the previous step we had nine zeros or nine ones. Nine zeros in the previous step is not possible as we have got nine zeros for the first time. Let's name the nine ones in the previous step as $b_1, b_2, ..., b_9$ in clockwise order and the bits in the step before the previous steps as $a_1, a_2..., a_9$ such that b_i was inserted between a_i and a_{i+1} (b_9 was inserted between a_9 and a_1). Clearly, a_i s should be alternation bits. But, if a_1 is 0, then a_9 will also be 0 and if a_1 is 1, then a_9 will also be 1. In both the cases b_9 cannot be 1. Hence, it is a contradiction. Therefore, we can never get nine zeros.

7. Write the numbers 1, 2, 3, ..., 2n on a board, where *n* is an odd integer. Pick any two of the numbers, *j* and *k*, write |j - k| on the board and erase *j* and *k*. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

Solution: For any odd integer *n*, the sum of integers from 1 to 2n is odd as (2n * (2n + 1))/2 is an odd number. We will now prove that the sum of integers will remain odd after every operation, that is, erasing two numbers *j* and *k* and writing |j - k|. This is sufficient to prove that the last integer will be an odd integer as when a single number is written on the board the sum of the numbers and the single number will have the same parity.

We will only analyse the two cases when both j and k are odd and j is odd and k is even. The other cases are very similar to them.

Case 1: Both j and k are odd

Let *sum* denote the sum of all the numbers before the operation. Then the sum of all the numbers after the operation is sum - (j + k) + |j - k| which is clearly an odd number as we are subtracting and adding even numbers to *sum*.

Case 2: j is an odd number and k is an even number

Again, let sum denote the sum of all the numbers before the operation. Then the sum of all the numbers after the operation is sum - (j + k) + |j - k| which is clearly an odd number as we are subtracting and adding odd numbers to sum.